Quantum mechanics II, Chapter 2 : What makes quantum different

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Sometimes observation kills.

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Problem 1 : The quantum eraser

1. Calculate what is observed at the screen when the polarization sheet is oriented at -45 degrees instead of +45 degrees. (In this case the screen only lets through $|\swarrow\rangle = \frac{1}{\sqrt{2}}(|H\rangle - |V\rangle)$ photons).

The projections of the incident states (with polarisations $|H\rangle$ and $|V\rangle$) upon the measurement outcomes ($|\swarrow\rangle$ and $|\nearrow\rangle$) are

$$\langle H|\nearrow\rangle = \langle H| \frac{1}{\sqrt{2}}(|H\rangle + |V\rangle) = \frac{1}{\sqrt{2}}, \qquad \langle H|\swarrow\rangle = \langle H| \frac{1}{\sqrt{2}}(|H\rangle - |V\rangle) = \frac{1}{\sqrt{2}}, \tag{1}$$

$$\langle V|\nearrow\rangle = \langle V| \frac{1}{\sqrt{2}}(|H\rangle + |V\rangle) = \frac{1}{\sqrt{2}}, \qquad \langle V|\swarrow\rangle = \langle V| \frac{1}{\sqrt{2}}(|H\rangle - |V\rangle) = -\frac{1}{\sqrt{2}}. \tag{2}$$

When transmitting only $|\nearrow\rangle$ photons, the states resulting from an incident $|H\rangle$ or $|V\rangle$ photon are identical. However, when transmitting only $|\swarrow\rangle$ photons, the resulting two states (corresponding to the right-column above) have a different sign :

$$|H\rangle \xrightarrow{\text{polarizer}} +|\swarrow\rangle \tag{3}$$

$$|V\rangle \xrightarrow{\text{polarizer}} -|\swarrow\rangle \tag{4}$$

$$\tag{5}$$

The post-polarizer state (normalised) reaching the screen is

$$|\Phi\rangle = \frac{1}{\sqrt{2}}\Big(\,|\psi_1\rangle - |\psi_2\rangle\,\Big)\,|\swarrow\rangle\,, \tag{6}$$

where $|\psi_1\rangle$ and $|\psi_2\rangle$ encode the position degree-of-freedom of the states resulting from transmission through slits 1 and 2 respectively. We can express these in the position basis in terms of continuously-parameterised amplitude $\psi_1(x) \in \mathbb{C}$, whereby

$$|\psi_1\rangle = \int \psi_1(x')\,|x'\rangle\,\mathrm{d}x', \tag{7}$$

$$\therefore \langle x|\psi_1\rangle = \langle x| \int \psi_1(x')\,|x'\rangle\,\mathrm{d}x' = \int \psi_1(x')\langle x|x'\rangle \mathrm{d}x' == \int \psi_1(x')\delta(x-x')\mathrm{d}x' = \psi_1(x), \tag{8}$$

and similarly for $|\psi_2\rangle$. Our post-polarizer state is ergo

$$|\Phi\rangle = \frac{1}{\sqrt{2}} \int \Big(\psi_1(x') - \psi_2(x')\Big)\,|x'\rangle\,|\swarrow\rangle\,\mathrm{d}x'. \tag{9}$$

and the probability density function of a photon incident on the screen at position x is :

$$p(x) = \langle\Phi| \Big(|x\rangle \langle x| \otimes \mathbb{1} \Big) |\Phi\rangle \tag{10}$$

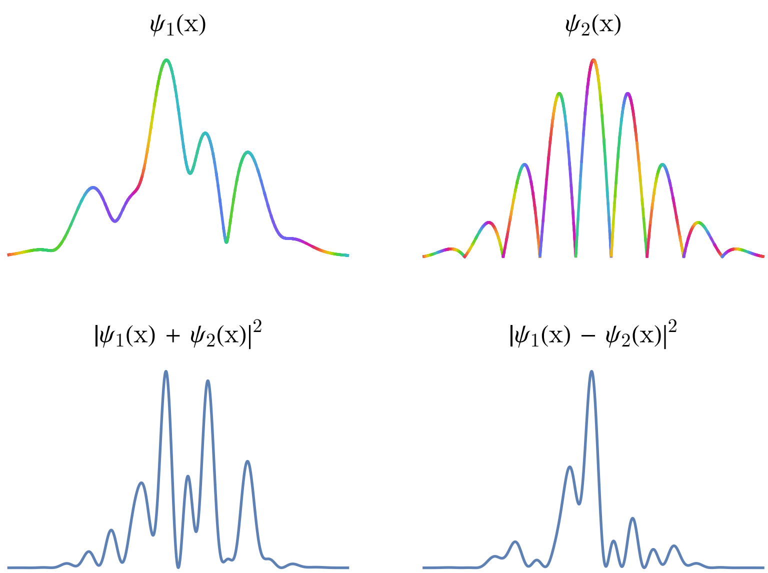
which, because $|\Phi\rangle$ is separable, mercifully simplifies to

$$p(x) = \left| \langle x| \frac{1}{\sqrt{2}} \int \Big(\psi_1(x') - \psi_2(x')\Big) |x'\rangle \, \mathrm{d}x' \right|^2 \tag{11}$$

$$= \frac{1}{2} \left|\psi_1(x) - \psi_2(x)\right|^2 . \tag{12}$$

Because the probability of a photon being admitted through the polarizer (i.e. collapses to $|\swarrow\rangle$ instead of $|\nearrow\rangle$) is $\frac{1}{2}$, we *could* multiply $p(x)$ by $\frac{1}{2}$ to quantify the probability (density function) of a photon *ever* reaching the screen at position x.

In any case, observe the sign of $\psi_2(x)$ is different from when the polarizer was oriented at $+45$ degrees. Depending on the wavefunctions $\psi_1(x)$ and $\psi_2(x)$, this can drastically change the observed interference pattern on the screen. For example, consider the below wavefunctions (where colour indicates complex phase) and their resulting interference patterns.



2. Consider now the system with no polarizers, where a photon passing through one of the slits flips the state of an atom from $|\downarrow\rangle$ to $|\uparrow\rangle$.

   (a) Calculate what is observed at the screen if the atoms are measured in basis $\{|\uparrow\rangle, |\downarrow\rangle\}$.

   (b) Calculate what is observed at the screen if the atoms are instead measured in basis $\{|\nearrow\rangle, |\swarrow\rangle\}$.

After passing the atom, our photons are described by state

$$|\Phi\rangle = \frac{1}{\sqrt{2}} \big(|\psi_1\rangle |\uparrow\rangle + |\psi_2\rangle |\downarrow\rangle \big) \tag{13}$$

If we measure in the basis $\{|\uparrow\rangle, |\downarrow\rangle\}$, we collapse the position degree-of-freedom to either state $|\psi_1\rangle$ or $|\psi_2\rangle$. The probability (density) of obtaining $|\uparrow\rangle$ atom measurement *and* seeing a photon incident at position x on the screen is

$$p_\uparrow(x) = \langle\Phi| \Big(|x\rangle \langle x| \otimes |\uparrow\rangle \langle\uparrow| \Big) |\Phi\rangle = \left| \frac{1}{\sqrt{2}} \langle x|\psi_1\rangle \right|^2 = \frac{1}{2} |\psi_1(x)|^2. \tag{14}$$

Conversely, obtaining a $|\downarrow\rangle$ atom measurement and seeing a photon at x has probability density

$$p_\downarrow(x) = \langle\Phi| \Big(|x\rangle \langle x| \otimes |\downarrow\rangle \langle\downarrow| \Big) |\Phi\rangle = \left| \frac{1}{\sqrt{2}} \langle x|\psi_2\rangle \right|^2 = \frac{1}{2} |\psi_2(x)|^2. \tag{15}$$

These yield the same patterns on the screen as we saw when we used a 90 degree polarization rotator behind one slit.

However, if we instead measure the atom in the basis $\{|\nearrow\rangle, |\swarrow\rangle\}$, we will find (skipping some algebra) that

$$p_\nearrow(x) = \langle\Phi| \Big(|x\rangle \langle x| \otimes |\nearrow\rangle \langle\nearrow| \Big) |\Phi\rangle = \frac{1}{4} |\psi_1(x) + \psi_2(x)|^2, \tag{16}$$

$$p_\swarrow(x) = \langle\Phi| \Big(|x\rangle \langle x| \otimes |\swarrow\rangle \langle\swarrow| \Big) |\Phi\rangle = \frac{1}{4} |\psi_1(x) - \psi_2(x)|^2. \tag{17}$$

This yields a distinct set of patterns on the screen.

3. Discuss whether this setup can be used for signalling.

Consider Bob performing measurements on the atom in either the $M_{\uparrow\downarrow} = \{|\uparrow\rangle, |\downarrow\rangle\}$ basis, or the $M_{\nearrow\swarrow} = \{|\nearrow\rangle, |\swarrow\rangle\}$ basis. He intends to signal a single bit to Alice (who observes the photons subsequently incident on the screen), encoded by his choice of measurement basis. That would require Alice can discern samples of probability density function

$$p_\uparrow(x) + p_\downarrow(x) = \frac{1}{2} \Big(|\psi_1(x)|^2 + |\psi_2(x)|^2 \Big)$$

from that of

$$p_\nearrow(x) + p_\swarrow(x) = \frac{1}{4} \Big(|\psi_1(x) + \psi_2(x)|^2 + |\psi_1(x) - \psi_2(x)|^2 \Big) \tag{18}$$

$$= \frac{1}{4} \left((\psi_1(x) + \psi_2(x))(\psi_1(x) + \psi_2(x))^* + (\psi_1(x) - \psi_2(x))(\psi_1(x) - \psi_2(x))^* \right) \tag{19}$$

which, after some tedious algebra, you will discover satisfies :

$$p_\nearrow(x) + p_\swarrow(x) = p_\uparrow(x) + p_\downarrow(x). \tag{20}$$

The probability of finding a photon incident at a particular x on the screen is independent of whether Bob measures $M_{\uparrow\downarrow}$ or $M_{\nearrow\swarrow}$. Ergo, the interference pattern is unchanged by Bob's measurement, and nothing is communicated. It is impossible to signal!

Problem 2 : No signalling

1. Suppose Alice and Bob share a two-qubit entangled state $\frac{\sqrt{3}}{2}|0\rangle_A|0\rangle_B + \frac{1}{2}|1\rangle_A|1\rangle_B$. Alice has the left qubit and Bob has the right. Alice attempts to communicate to Bob by measuring her qubit in the Z or X eigenbases.

   Compute the probability that Alice obtains each outcome, and the corresponding collapsed states of Bob's qubit.

Let $|\psi\rangle_{AB} = \frac{\sqrt{3}}{2}|00\rangle + \frac{1}{2}|11\rangle$. The probabilities that Alice obtains each outcome when measuring in the Z basis are :

$$p(0) = \langle\psi|_{AB}(|0\rangle\langle 0|_A \otimes \mathbb{1}_B)|\psi\rangle_{AB} = \frac{3}{4}, \tag{21}$$

$$p(1) = \langle\psi|_{AB}(|1\rangle\langle 1|_A \otimes \mathbb{1}_B)|\psi\rangle_{AB} = \frac{1}{4}. \tag{22}$$

These outcomes respectively collapse Bob's state to :

$$|\psi_0\rangle = |0\rangle \tag{23}$$

$$|\psi_1\rangle = |1\rangle \tag{24}$$

The probabilities of Alice's outcomes when measuring in the X basis are :

$$p(+) = \langle\psi|_{AB}(|+\rangle\langle +|_A \otimes \mathbb{1}_B)|\psi\rangle_{AB} = \frac{1}{2} \tag{25}$$

$$p(-) = \langle\psi|_{AB}(|-\rangle\langle -|_A \otimes \mathbb{1}_B)|\psi\rangle_{AB} = \frac{1}{2}, \tag{26}$$

and Bob's corresponding collapsed states are

$$|\phi_0\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle \tag{27}$$

$$|\phi_1\rangle = \frac{\sqrt{3}}{2}|0\rangle - \frac{1}{2}|1\rangle \tag{28}$$

2. Imagine Bob subsequently performs a measurement. We seek to show that the probability of him obtaining an outcome corresponding to a completely arbitrary one-qubit projector $\hat{\Pi}$ is the same regardless of whether Alice measured X or Z. That would mean he cannot locally distinguish what measurement Alice implemented, and so cannot determine her message.

   Write down the probability that Bob obtains outcome Π if Alice measures Z or X respectively, i.e. $P_B(\Pi|A \text{ measures } Z)$ and $P_B(\Pi|A \text{ measures } X)$. Hence show that $P_B(\Pi|A \text{ measures } Z) = P_B(\Pi|A \text{ measures } X)$.

Let Π notate Bob's measurement outcome with corresponding projector $\hat{\Pi}$. Using the law of total probability, we know that

$$P(B \text{ obtains } \Pi | A \text{ measures } Z) = \quad P(B \text{ obtains } \Pi \text{ and } A \text{ obtains } 0 | A \text{ measures } Z) \tag{29}$$
$$+ P(B \text{ obtains } \Pi \text{ and } A \text{ obtains } 1 | A \text{ measures } Z) \tag{30}$$
$$= \frac{3}{4}\langle 0|\hat{\Pi}|0\rangle + \frac{1}{4}\langle 1|\hat{\Pi}|1\rangle. \tag{31}$$

Similarly,

$$P(B \text{ obtains } \hat{\Pi} | A \text{ measures } X) = \frac{1}{2}\langle \phi_0|\hat{\Pi}|\phi_0\rangle + \frac{1}{2}\langle \phi_1|\hat{\Pi}|\phi_1\rangle \tag{32}$$
$$= \frac{1}{2}\left(\frac{\sqrt{3}}{2}\langle 0| + \frac{1}{2}\langle 1|\right)\hat{\Pi}\left(\frac{\sqrt{3}}{2}|0\rangle + |1\rangle\, 2\,\langle 1|\right) \tag{33}$$
$$+ \frac{1}{2}\left(\frac{\sqrt{3}}{2}\langle 0| - \frac{1}{2}\langle 1|\right)\hat{\Pi}\left(\frac{\sqrt{3}}{2}|0\rangle - |1\rangle\, 2\,\langle 1|\right) \tag{34}$$
$$= \frac{3}{4}\langle 0|\hat{\Pi}|0\rangle + \frac{1}{4}\langle 1|\hat{\Pi}|1\rangle \tag{35}$$
$$= P(B \text{ obtains } \Pi | A \text{ measures } Z) \tag{36}$$

We have shown that Bob's measurement statistics are entirely independent of whether Alice measures in the X or Z basis. Alice ergo cannot communicate *any* information to Bob via her measurement basis, *super*luminal or otherwise!

3. Consider now that Alice and Bob share an arbitrary two-qubit entangled state $|\phi_{AB}\rangle$. Show that Alice cannot superluminally communicate a bit to Bob by performing either X or Z measurements on her qubit.

Let's assume Alice and Bob each have a qubit of a completely general two-qubit state;

$$|\psi\rangle = \alpha\,|00\rangle + \beta\,|01\rangle + \delta\,|10\rangle + \gamma\,|11\rangle\,, \tag{37}$$

where the greek symbols are normalised complex amplitudes. This general state includes all possible entangled states, and all separable ones too (which we obviously know cannot be used to signal). We can demonstrate the impossibility of communication via exactly our method in the previous section, now using these symbolic amplitudes!

Alice measuring her qubit in the Z basis has probabilities of outcomes :

$$P(A \text{ obtains } 0 | A \text{ measures } Z) = |\alpha|^2 + |\beta|^2, \tag{38}$$
$$P(A \text{ obtains } 1 | A \text{ measures } Z) = 1 - |\alpha|^2 - |\beta|^2 \tag{39}$$
$$= |\delta|^2 + |\gamma|^2. \tag{40}$$

We simply read off these probabilities using the Born rule, because our measurement basis was the same basis as our state. These scenarios correspond to respective post-measurement (renormalised) states :

$$|\psi_{A=0}\rangle = \frac{1}{\sqrt{|\alpha|^2 + |\beta|^2}}\left(\alpha\,|00\rangle + \beta\,|01\rangle\right), \qquad |\psi_{A=1}\rangle = \frac{1}{\sqrt{|\delta|^2 + |\gamma|^2}}\left(\delta\,|10\rangle + \gamma\,|11\rangle\right). \tag{41}$$
$$|\phi_{A=0}\rangle = \frac{1}{\sqrt{|\alpha|^2 + |\beta|^2}}\left(\alpha\,|0\rangle + \beta\,|1\rangle\right), \qquad |\phi_{A=1}\rangle = \frac{1}{\sqrt{|\delta|^2 + |\gamma|^2}}\left(\delta\,|0\rangle + \gamma\,|1\rangle\right). \tag{42}$$

which we have separated from Bob's resulting states $|\phi_{A=i}\rangle$, where $|\psi_{A=i}\rangle = |i\rangle |\phi_{A=i}\rangle$.

We now let Π notate the outcome of Bob's subsequent, arbitrary one-qubit measurement, with corresponding projector $\hat{\Pi}$. Again, the law of total probability, together with Bayes' theorem, states

$$\begin{aligned}
&P(B \text{ obtains } \Pi | A \text{ measures } Z)\\
= \quad & P(B \text{ obtains } \Pi \text{ and } A \text{ obtains } 0 | A \text{ measures } Z)\\
+ \quad & P(B \text{ obtains } \Pi \text{ and } A \text{ obtains } 1 | A \text{ measures } Z)\\
= \quad & P(A \text{ obtains } 0 \; | A \text{ measures } Z) \times P(B \text{ obtains } \Pi | A \text{ measures } Z = 0)\\
+ \quad & P(A \text{ obtains } 1 \; | A \text{ measures } Z) \times P(B \text{ obtains } \Pi | A \text{ measures } Z = 1)\\
= & \left(|\alpha|^2 + |\beta|^2\right) |\langle \Pi | \phi_{A=0} \rangle|^2 + \left(|\delta|^2 + |\gamma|^2\right) |\langle \Pi | \phi_{A=1} \rangle|^2\\
= & | \langle \Pi | \left(\alpha |0\rangle + \beta |1\rangle\right)|^2 + | \langle \Pi | \left(\delta |0\rangle + \gamma |1\rangle\right)|^2\\
= & (|\alpha|^2 + |\delta|^2)\cos^2\theta + (|\beta|^2 + |\gamma|^2)\sin^2\theta + (\alpha\beta^* + \delta\gamma^*)\cos\theta\sin\theta e^{i\phi} + (\alpha^*\beta + \delta^*\gamma)\cos\theta\sin\theta e^{-i\phi}
\end{aligned}$$

where we have expressed outcome state $|\Pi\rangle = \cos\theta |0\rangle + \sin\theta e^{-i\phi} |1\rangle$ as a generate state in terms of $\theta, \phi \in \mathbb{R}$.

Before we study the outcomes and probabilities of Alice measuring in the X basis, let's first express the left qubit in the $\{|\pm\rangle\}$ basis :

$$\begin{aligned}
|\psi\rangle = & \frac{\alpha}{\sqrt{2}} \left(|+\rangle + |-\rangle\right) |0\rangle + \frac{\beta}{\sqrt{2}} \left(|+\rangle + |-\rangle\right) |1\rangle && (43)\\
& + \frac{\delta}{\sqrt{2}} \left(|+\rangle - |-\rangle\right) |0\rangle + \frac{\gamma}{\sqrt{2}} \left(|+\rangle - |-\rangle\right) |1\rangle && (44)\\
= & \frac{1}{\sqrt{2}}(\alpha + \delta) |+\rangle |0\rangle + \frac{1}{\sqrt{2}}(\beta + \gamma) |+\rangle |1\rangle + && (45)\\
& \frac{1}{\sqrt{2}}(\alpha - \delta) |-\rangle |0\rangle + \frac{1}{\sqrt{2}}(\beta - \gamma) |-\rangle |1\rangle && (46)
\end{aligned}$$

For convenience, we will define $a, b, c, d \in \mathbb{C}$ as

$$\begin{aligned}
a = & \frac{1}{\sqrt{2}}(\alpha + \delta) && (47)\\
b = & \frac{1}{\sqrt{2}}(\beta + \gamma) && (48)\\
c = & \frac{1}{\sqrt{2}}(\alpha - \delta) && (49)\\
d = & \frac{1}{\sqrt{2}}(\beta - \gamma). && (50)
\end{aligned}$$

to re-express our general state as

$$|\psi\rangle = a |+\rangle |0\rangle + b |+\rangle |1\rangle + c |-\rangle |0\rangle + d |-\rangle |1\rangle . \qquad (51)$$

These new variables will later spare us from some tedious algebra.

Let's now consider when Alice measures in the X basis, obtaining outcomes $\pm$. Because we have expressed Alice's qubit in $|\psi\rangle$ in the $\pm$ basis, we can again simply read off the probabilities using the Born rule.

$$P(A \text{ obtains } +|A \text{ measures } X) = |a|^2 + |b|^2 \tag{52}$$
$$P(A \text{ obtains } -|A \text{ measures } X) = 1 - P(A \text{ obtains } +|A \text{ measures } X) \tag{53}$$
$$= 1 - |a|^2 + |b|^2 = |c|^2 + |d|^2 \tag{54}$$

These correspond to collapsed states :

$$|\psi_{A=+}\rangle = \frac{1}{\sqrt{|a|^2 + |b|^2}} \left(a\,|+,0\rangle + b\,|+,1\rangle\right), \qquad |\psi_{A=-}\rangle = \frac{1}{\sqrt{|c|^2 + |d|^2}} \left(c\,|-,0\rangle + d\,|-,1\rangle\right). \tag{55}$$
$$|\phi_{A=+}\rangle = \frac{1}{\sqrt{|a|^2 + |b|^2}} \left(a\,|0\rangle + b\,|1\rangle\right), \qquad |\phi_{A=-}\rangle = \frac{1}{\sqrt{|c|^2 + |d|^2}} \left(c\,|0\rangle + d\,|1\rangle\right). \tag{56}$$

The total law of probability, and subsequent Bayes' theorem, state that

$$\begin{aligned}
P(B \text{ obtains } \Pi|A \text{ measures } X) =& P(B \text{ obtains } \Pi \text{ and } A \text{ obtains } +|A \text{ measures } X)\\
&+ P(B \text{ obtains } \Pi \text{ and } A \text{ obtains } -|A \text{ measures } X)\\
=& P(A \text{ obtains } +\,|A \text{ measures } X) \times P(B \text{ obtains } \Pi|A \text{ measures } X = +)\\
&+ P(A \text{ obtains } -\,|A \text{ measures } X) \times P(B \text{ obtains } \Pi|A \text{ measures } X = -)\\
=& (|a|^2 + |c|^2)\cos^2\theta + (|b|^2 + |d|^2)\sin^2\theta\\
&+ (ab^* + cd^*)\cos\theta\sin\theta e^{i\phi} + (a^*b + c^*d)\cos\theta\sin\theta e^{-i\phi}
\end{aligned}$$

Substituting $\alpha, \beta, \delta, \gamma$ into our definitions of a, b, c, d constrains Trivially

$$|a|^2 + |c|^2 = |\alpha|^2 + |\delta|^2 \tag{57}$$
$$|b|^2 + |d|^2 = |\beta|^2 + |\gamma|^2 \tag{58}$$
$$ab^* + cd^* = \frac{1}{2}(\alpha + \delta)(\beta^* + \gamma^*) - \frac{1}{2}(\alpha + \delta)(\beta^* - \gamma^*) \ = \alpha\beta^* + \delta\gamma^* \tag{59}$$

which, substituted into our previous probability, yields

$$P(B \text{ obtains } \Pi|A \text{ measures } X) \tag{60}$$
$$= (|a|^2 + |c|^2)\cos^2\theta + (|b|^2 + |d|^2)\sin^2\theta + (ab^* + cd^*)\cos\theta\sin\theta e^{i\phi} + (a^*b + c^*d)\cos\theta\sin\theta e^{-i\phi} \tag{61}$$
$$= (|\alpha|^2 + |\delta|^2)\cos^2\theta + (|\beta|^2 + |\gamma|^2)\sin^2\theta + (\alpha\beta^* + \delta\gamma^*)\cos\theta\sin\theta e^{i\phi} + (\alpha^*\beta + \delta^*\gamma)\cos\theta\sin\theta e^{-i\phi} \tag{62}$$
$$= P(B \text{ obtains } \Pi|A \text{ measures } Z) \tag{63}$$

which we recognise (painfully) to be the same probability when Alice measured in the Z basis. Ergo, even when using *any* two-qubit state and *any* measurement, Bob still cannot infer Alice's measurement axis. Alice thus cannot communicate at all via her measurement, superluminally or otherwise!

Problem 3 : Bell inequality (quantum psychics version)

Every time Alice and Bob get told 'H' they must give opposite answers, but otherwise they must give the same answer.

Alice and Bob's strategy :

— If Alice gets told 'H' she measures in the Z basis and says 'Y' if she gets '$|0\rangle$' and 'N' if she gets '$|1\rangle$'.

— If Alice gets told 'T' she measures in the X basis and says 'Y' if she gets '$|+\rangle$' and 'N' if she gets '$|-\rangle$'.

— If Bob gets told 'H' he measures in the basis

$$\{|h\rangle = \sin(\pi/8)|0\rangle + \cos(\pi/8)|1\rangle\ ,\ |\overline{h}\rangle = \cos(\pi/8)|0\rangle - \sin(\pi/8)|1\rangle\} \tag{64}$$

and says 'Y' if he gets '$|h\rangle$' and 'N' if she gets '$|\overline{h}\rangle$'.

— If Bob gets told 'T' he measures in the basis

$$\{|t\rangle = \cos(\pi/8)|0\rangle + \sin(\pi/8)|1\rangle\ ,\ |\overline{t}\rangle = \sin(\pi/8)|0\rangle - \cos(\pi/8)|1\rangle\} \tag{65}$$

and says 'Y' if he gets '$|t\rangle$' and 'N' if she gets '$|\overline{t}\rangle$'.

From this test, we can write the probability of winning as follows

$$\begin{aligned} P_{\text{win}} = \frac{1}{4}\Big(& P(A_H = 1, B_H = -1|H,H) + P(A_H = -1, B_H = 1|H,H) \\ & + P(A_H = 1, B_T = 1|H,T) + P(A_H = -1, B_T = -1|H,T) \\ & + P(A_T = 1, B_H = 1|T,H) + P(A_T = -1, B_H = -1|T,H) \\ & + P(A_T = 1, B_T = 1|T,T) + P(A_T = -1, B_T = -1|T,T)\Big) \end{aligned} \tag{66}$$

To show that $P_{\text{Quantum}} = \cos(\pi/8)^2$ we need to compute these eight that we have in P_{win} while we consider that Alice and Bob share entangled Bell states, $|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Using Alice and Bob's strategy to do the game the terms in P_{win} are as follows.

For $P(A_H = 1, B_H = -1|H,H)$ from the strategies, Alice measures in the Z basis and gets $|0\rangle$ and Bob measures in $\{|h\rangle, |\overline{h}\rangle\}$ basis and gets $|\overline{h}\rangle$.

$$\begin{aligned} P(A_H = 1, B_H = -1|H,H) &= \langle\phi^+||0\rangle\langle 0| \otimes |\overline{h}\rangle\langle\overline{h}|\,\Big|\phi^+\Big\rangle \\ &= \frac{1}{2}\cos^2(\frac{\pi}{8}) \end{aligned} \tag{67}$$

For $P(A_H = -1, B_H = 1|H,H)$ from the strategies, Alice measures in the Z basis and gets $|1\rangle$ and Bob measures in $\{|h\rangle, |\overline{h}\rangle\}$ basis and gets $|h\rangle$.

$$\begin{aligned} P(A_H = -1, B_H = 1|H,H) &= \langle\phi^+||1\rangle\langle 1| \otimes |h\rangle\langle h|\,\Big|\phi^+\Big\rangle \\ &= \frac{1}{2}\cos^2(\frac{\pi}{8}) \end{aligned} \tag{68}$$

For $P(A_H = 1, B_T = 1|H,T)$ from the strategies, Alice measures in the Z basis and gets $|0\rangle$ and Bob measures in $\{|t\rangle, |\bar{t}\rangle\}$ basis and gets $|t\rangle$.

$$\begin{aligned} P(A_H = 1, B_H = 1|H,T) &= \langle\phi^+||0\rangle\langle 0| \otimes |t\rangle\langle t| \left|\phi^+\right\rangle \\ &= \frac{1}{2}\cos^2(\frac{\pi}{8}) \end{aligned} \tag{69}$$

For $P(A_H = -1, B_T = -1|H,T)$ from the strategies, Alice measures in the Z basis and gets $|1\rangle$ and Bob measures in $\{|t\rangle, |\bar{t}\rangle\}$ basis and gets $|\bar{t}\rangle$.

$$\begin{aligned} P(A_H = -1, B_H = -1|H,T) &= \langle\phi^+||1\rangle\langle 1| \otimes |\bar{t}\rangle\langle \bar{t}| \left|\phi^+\right\rangle \\ &= \frac{1}{2}\cos^2(\frac{\pi}{8}) \end{aligned} \tag{70}$$

For $P(A_T = 1, B_H = 1|T,H)$ from the strategies, Alice measures in the X basis and gets $|+\rangle$ and Bob measures in $\{|h\rangle, |\bar{h}\rangle\}$ basis and gets $|h\rangle$.

$$\begin{aligned} P(A_H = 1, B_H = 1|T,H) &= \langle\phi^+||+\rangle\langle +| \otimes |h\rangle\langle h| \left|\phi^+\right\rangle \\ &= \frac{1}{4}(1+\sin\left(\frac{\pi}{4}\right)) = \frac{1}{2}\cos^2(\frac{\pi}{8}) \end{aligned} \tag{71}$$

For $P(A_T = -1, B_H = -1|T,H)$ from the strategies, Alice measures in the X basis and gets $|-\rangle$ and Bob measures in $\{|h\rangle, |\bar{h}\rangle\}$ basis and gets $|\bar{h}\rangle$.

$$\begin{aligned} P(A_H = -1, B_H = -1|T,H) &= \langle\phi^+||-\rangle\langle -| \otimes |\bar{h}\rangle\langle \bar{h}| \left|\phi^+\right\rangle \\ &= \frac{1}{4}(1+\sin\left(\frac{\pi}{4}\right)) = \frac{1}{2}\cos^2(\frac{\pi}{8}) \end{aligned} \tag{72}$$

For $P(A_T = 1, B_T = 1|T,T)$ from the strategies, Alice measures in the X basis and gets $|+\rangle$ and Bob measures in $\{|t\rangle, |\bar{t}\rangle\}$ basis and gets $|t\rangle$.

$$\begin{aligned} P(A_T = 1, B_T = 1|T,T) &= \langle\phi^+||+\rangle\langle +| \otimes |t\rangle\langle t| \left|\phi^+\right\rangle \\ &= \frac{1}{4}(1+\sin\left(\frac{\pi}{4}\right)) = \frac{1}{2}\cos^2(\frac{\pi}{8}) \end{aligned} \tag{73}$$

For $P(A_T = -1, B_T = -1|T,T)$ from the strategies, Alice measures in the X basis and gets $|-\rangle$ and Bob measures in $\{|t\rangle, |\bar{t}\rangle\}$ basis and gets $|\bar{t}\rangle$.

$$\begin{aligned} P(A_T = -1, B_T = -1|T,T) &= \langle\phi^+||-\rangle\langle -| \otimes |\bar{t}\rangle\langle \bar{t}| \left|\phi^+\right\rangle \\ &= \frac{1}{4}(1+\sin\left(\frac{\pi}{4}\right)) = \frac{1}{2}\cos^2(\frac{\pi}{8}) \end{aligned} \tag{74}$$

Finally, we have that Alice and Bob can win the test with probability

$$P_{\text{Quantum}} = \cos(\pi/8)^2 = \frac{2+\sqrt{2}}{4} \approx 0.854\,. \tag{75}$$